EFFECTIVE ELECTRICAL CHARACTERISTICS OF A CONDUCTING MEDIUM WITH SPHERICAL INCLUSIONS THAT ARE ANISOTROPIC AS A CONSEQUENCE OF THE HALL EFFECT

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The problem of the field and current distribution in a conducting continuum containing spherical inclusions that display the Hall effect is solved. An equation for the effective conductivity tensor of such a medium is obtained and studied. It is assumed that the concentration of inclusions is low.

As a consequence of the Hall effect, the conductivity of a medium in the case of a plasma and semiconductors in a magnetic field is described by a tensor which, in the three-dimensional case, has the form

$$\hat{\sigma} = \sigma_\mu \begin{pmatrix} 1 & -\beta & 0 \\ \beta & 1 & 0 \\ 0 & 0 & \sigma/\sigma_\mu \end{pmatrix}, \quad \sigma_\mu = \sigma/(1+\beta^2),$$

where σ is scalar conductivity and β is the Hall parameter. A magentic field $\mathbf{H}$ is directed along the z axis $[\mathbf{H} = (0, 0, \mathbf{H})]$.

We arrive at the equations

$$\nabla \cdot \mathbf{j} = 0, \quad \nabla \times \mathbf{E} = 0, \quad j = \hat{\sigma}\mathbf{E},$$

describing the field and current distribution, disregarding induced magnetic fields. The usual continuity conditions of the current component normal to the boundary and the tangential component of the electric field are satisfied at the boundary of the heterogeneous medium.

The problem of the field and current distributions in the region of a spherical heterogeneous inclusion has an exact solution whenever the master phase conductivity is a scalar σ_1 and the conductivity of the inclusions is a tensor $\hat{\sigma}_2$.

It is convenient to introduce the electric potentials $\varphi_{1,2}(\mathbf{E}_{1,2} = -\nabla\varphi_{1,2})$ in order to solve the problem by means of a system of equations. All physical constants of the master phase and of the inclusions, respectively, are defined here and below by the subscripts "1" and "2".

When a uniform electric field $\mathbf{E}_1(\infty)$ is defined at infinity, the general solution of the problem for a sphere of radius R situated at the coordinate origin has the form

$$\varphi_1 = -\mathbf{E}_1(\infty) \cdot \mathbf{r} + (\mathbf{A} \cdot \mathbf{r})/r^3, \quad \varphi_2 = \mathbf{B} \cdot \mathbf{r},$$

where $\mathbf{A}$ and $\mathbf{B}$ are constants determined from the boundary conditions.

Let us present equations for the electric field $\mathbf{E}_2$ and current $\mathbf{j}_2$ in the region of an inclusion that will be henceforth necessary in order to determine the effective conductivity tensor:

$$j_{2x} = \frac{3\sigma_2}{(\sigma_2 + 2\sigma_1)(1 + \mathrm{tg}^2\,\pi\varepsilon)} (j_{1x}(\infty) - \mathrm{tg}\,\pi\varepsilon j_{1y}(\infty)), \tag{1}$$

$$j_{2y} = \frac{3\sigma_2}{(\sigma_2 + 2\sigma_1)(1 + \mathrm{tg}^2\,\pi\varepsilon)} (\mathrm{tg}\,\pi\varepsilon j_{1x}(\infty) + j_{1y}(\infty)),$$

$$j_{2z} = \frac{3\sigma_2}{\sigma_2 + 2\sigma_1} j_{1z}(\infty), \quad \mathbf{j}_2 = \hat{\sigma}_2\mathbf{E}_2, \quad \mathbf{j}_1 = \sigma_1\mathbf{E}_1.$$

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The reduced Hall angle $\pi\varepsilon$ is determined by the equation

$$\pi\varepsilon = \text{arctg}\ [2\beta_2\sigma_1/(\sigma_2+2\sigma_1)]$$

and is the angle between the current components j_{2x} and j_{2y} ($\pi\varepsilon = \arctan j_{2y}/j_{2x}$) whenever a field $\mathbf{E}_1(\infty)$ of the form

$$E_{1x}=E_{1x}(\infty),\ E_{1y}=0,\ E_{1z}=0 \text{ when } r\to\infty.$$

is defined at infinity.

The constant $\mathbf{A}$ necessary for determining the field beyond the inclusion has the form

$$A=R^3(\mathbf{E}_1(\infty)-\mathbf{E}_2).$$

The field and current distribution found in Eq. (1) allows us to determine the effective conductivity tensor $\hat{\sigma}_{\text{eff}}$ connecting the voume-averaged electric field strength $\mathbf{E}_{\text{eff}}$ and the electric current density $\mathbf{I}_{\text{eff}}$,

$$\mathbf{I}_{eff}=\hat{\sigma}_{eff}\mathbf{E}_{eff}.$$

Let us consider the volume-averaged integral

$$\frac{1}{V}\int_V (\mathbf{j}-\sigma_1\mathbf{E})\,dV = \mathbf{I}_{eff}-\sigma_1\mathbf{E}_{eff}$$

and take into account the fact that a field equal to the effective field is defined over inclusions in the medium at infinity, arriving at an equation for the effective conductivity tensor,

$$\hat{\sigma}_{eff}=\sigma_1\delta_{ik}+3c\sigma_1\begin{Vmatrix} \dfrac{\Delta-\frac{1}{2}\text{tg}^2\pi\varepsilon}{1+\text{tg}^2\pi\varepsilon}, & -\dfrac{3}{2}\dfrac{\text{tg}\,\pi\varepsilon}{1+\text{tg}^2\pi\varepsilon}\dfrac{1}{1+2\frac{\sigma_1}{\sigma_2}}, & 0 \\ \dfrac{3}{2}\dfrac{\text{tg}\,\pi\varepsilon}{1+\text{tg}^2\pi\varepsilon}\dfrac{1}{1+2\frac{\sigma_1}{\sigma_2}}, & \dfrac{\Delta-\frac{1}{2}\text{tg}^2\pi\varepsilon}{1+\text{tg}^2\pi\varepsilon}, & 0 \\ 0, & 0, & \Delta \end{Vmatrix}, \qquad (2)$$

where $(\sigma_2-\sigma_1)/(\sigma_2+2\sigma_1)$ $(-{}^1/_2\le\Delta\le1)$ is the relative fluctuation of conductivity, c is the concentration of connections, and δ_{ik} is the Kronecker symbol.

It is of interest to determine the limits of applicability of Eq. (2) as a function of c. This cannot be done in the general case, though, according to previous results [1], $c<0.25$ for the tensor component (2) along the magnetic field (and thereby for the components across the magnetic field).

It is clear from Eq. (2) that in the magentic field, the magnetic conductivity of the medium as a whole becomes a tensor due to the conductivity and anisotropy of the inclusions even with a low concentration of inclusions.

In analyzing Eq. (2), it is convenient to consider an additional term $\hat{\sigma}_d=\hat{\sigma}_{\text{eff}}-\sigma_1\delta_{ik}$, which contains a component $\sigma_d=3c\sigma_1\Delta$, independent of the magnetic field, and components that are functions of the magnetic field. We may introduce the effective conductivity σ_{de} and the effective Hall parameter β_{de} for $\hat{\sigma}_d$ in a plane normal to $\mathbf{H}$,

$$\sigma_{de}=3c\sigma_1\left(\Delta-\frac{1}{2}\text{tg}^2\pi\varepsilon\right)(1+\beta_{de}^2)/(1+\text{tg}^2\pi\varepsilon),$$

$$\beta_{de}=\frac{3}{2}\text{tg}\,\pi\varepsilon\Big/\left(\Delta-\frac{1}{2}\text{tg}^2\pi\varepsilon\right)\left(1+2\frac{\sigma_1}{\sigma_2}\right).$$

The "additional" current through the inclusions and associated with σ_{de} and β_{de} is determined by the equations

$$\mathbf{I}_d=\sigma_d\mathbf{E}_1(\infty);\ \sigma_d=\sigma_{de}/(1-i\beta_{de});$$

$$\mathbf{I}_d=I_{dx}+iI_{dy};\ \mathbf{E}_1(\infty)=E_{1x}(\infty)+iE_{1y}(\infty);\ i^2=-1.$$

The asymptotic formulas for field and current in a strong magnetic field ($\beta_2\gg1$) have the form

$$E_{2x} = \frac{3}{2} E_{1x}(\infty),\ E_{2y} = \frac{3}{2} E_{1y}(\infty),\ E_{2z} = \frac{3\sigma_1}{\sigma_2 + 2\sigma_1} E_{1z}(\infty),$$
$$j_{2x} = -\frac{3}{2}\frac{\sigma_2}{\beta_2} E_{1y}(\infty),\ j_{2y} = \frac{3}{2}\frac{\sigma_2}{\beta_2} E_{1x}(\infty),\ j_{2z} = \frac{3\sigma_1\sigma_2}{\sigma_2 + 2\sigma_1} E_{1z}(\infty), \tag{3}$$

while the effective conductivity tensor (2) (terms on the order of $1/\beta_2$ are preserved) is given by

$$\hat{\sigma}_{eff} = \sigma_1\delta_{ik} - \frac{3}{2} c\sigma_1 \begin{pmatrix} 1, & \frac{3}{2}\frac{\sigma_2}{\sigma_1}\frac{1}{\beta_2}, 0 \\ -\frac{3}{2}\frac{\sigma_2}{\sigma_1}\frac{1}{\beta_2}, & 1,\ 0 \\ 0, & 0,\ 2\Delta \end{pmatrix}. \tag{4}$$

It is clear from Eqs. (3) and (4) that in a strong magnetic field, an anisotropic medium with spherical anisotropic inclusions possesses the unusual property that inclusions with arbitrary conductivity exert the same influence as do weakly conducting ($\sigma_2 \ll \sigma_1$) inclusions in the absence of a magnetic field. An analogous situation occurs in a two-dimensional model with the disk inclusions [2].

In our problem, β_{de} decreases with increasing magnetic field,

$$\beta_{de} = 3\sigma_2/2\sigma_1\beta_2,$$

while the parameter σ_{de} reaches saturation in $\beta_3[\sigma_{de} = -(3/2)c\sigma_1]$. No effective anisotropy is manifested along the magnetic field. In this direction, the medium is isotropic with isotropic inclusions [3].

LITERATURE CITED

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